A New Treatise by al-Kāshī on the Depression of the Visible Horizon

by MOHAMMAD BAGHERI

In this paper I present an edited Arabic text and an English translation of a short treatise, wrongly attributed to Bābā Afḍal al-Dīn al-Kāshī (7th/13th century), an Iranian mystic poet who lived in Kāshān, Iran. The treatise is about the determination of the visible horizon, a geometrical problem mentioned in two letters by Ghiyāth al-Dīn Jamshīd al-Kāshī (ca. 826/1422) to his father. I discuss the history of this problem and I show as well that the short treatise was written not by Bābā Afḍal but most likely by Jamshīd al-Kāshī.

Ghiyāth al-Dīn Jamshīd al-Kāshī, an eminent Iranian mathematician and astronomer of the 15th century, left his birthplace Kāshān for Samarkand in A.D. 1421 and joined the scientific circle of Ulugh Beg. From Samarkand he wrote letters in Persian to his father who lived in Kāshān and who apparently was familiar with mathematics and astronomy. In these letters, al-Kāshī described — among other things — the scientific problems which were discussed in the scientific meetings in Samarkand. One of these letters has been known since 1859, when a learned Qājār prince, I'tiḍād al-Salṭana, quoted a fragment of it in the report of his visit to the Maragha observatory in the company of the Qājār king Nāṣir al-Dīn. This letter has been published several times, and translations into English, Turkish, Arabic, Russian and Uzbek have also appeared with introductions and commentaries. Another letter was found recently. The text was published in my book *Az Samarqand be Kāshān: Nāmehāye Ghiyāth al-Dīn Jamshīd Kāshānī be pedarash* (From Samarkand to Kāshān: Letters of al-Kāshī to His Father), [Bagheri 1996]; and an English translation with an introduction and commentary can be found in [Bagheri 1997].

The problem of the depression of the visible horizon is mentioned in both letters among several other problems brought up by Ulugh Beg. This problem remained unsolved in Samarkand until al-Kāshī arrived there and managed to solve it. In the letter known since 1859, the problem is described as follows:

> Suppose that a man stands at a spot around which the curvature of the surface of the earth is quite accurate; let his height be three and a half *gaz,* by the *dhirā' of the arm* [i.e., 3.5 cubits]. At what distance will a visual ray, leaving his eyes and touching the surface of the earth tangentially, intersect with the true horizon and with what angle of depression will it reach the outermost sphere of the firmament? [Sayılı 1960: 97–98]

Al-Kāshī found it an easy problem and solved it completely in one day. Sayılı gives no geometrical explanation, but he says, "In the problem concerning the visual ray and the true horizon, the difficulty encountered probably concerned calculation and

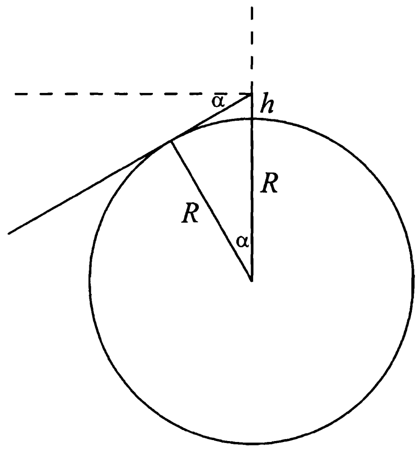


Figure 1

the derivation of numerical results with a given degree of accuracy" [Sayılı 1960: 47]. Prof. E.S. Kennedy, who also published an English translation of this letter, provides the following explanation of the problem without providing any figure: "If R is the radius of the earth and h the height of the person (measured in the same units), then the angular distance of the true horizon is arccos $[R/(R + h)]$, and this angle plus a quadrant is the true horizon's depression measured from the zenith" [Kennedy 1960: 208] (see Fig. 1).

In his commentary to the Russian translation of this letter, G. Sabirov gives another interpretation of the problem as shown in Fig. 2, where he calculates the complement of the angle which Kennedy calculated. Sabirov gives the formulas for the angle and distance in question as

$$\alpha = \arcsin\frac{R}{R+h} \qquad \text{and} \qquad l = \sqrt{h(2R+h)}$$

[Sabirov 1973: 204–205]. In the other editions and translations of this letter, there is no reference to what al-Kāshī may have had in mind regarding this problem.

The discovery of the other letter from al-Kāshī to his father sheds more light on this problem. In this letter, he writes:

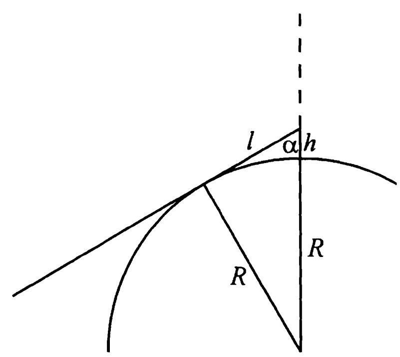


Figure 2

> During the period that the arena was theirs [his rivals' in Samarkand who wanted to test him], in the discussions held in the presence of His Royal Majesty [Ulugh Beg] they were confronted with some difficulties into which they had looked for a month or two or even for a year, but to which no solution had been found. For example, this problem: [Let us suppose] somebody is standing on a perfectly circular ground or on the sea surface, and the visual ray issuing from his eyes is tangent to that, and [then] reaches the sphere of the ecliptic [*falak al-burūj*, i.e., the sphere of the fixed stars[1]]. Now at which distance will [that ray] intersect the true horizon, and, where it reaches the sphere of the ecliptic, how much will it

1 This expression for the sphere of the fixed stars is found in the section on astronomical terminology in the commentary of Bīrjandī (d. A.D. 1527/8) on *Zīj-i Ulugh Beg* (Tehran, Sipahsalar Library, MS 680, fol. 61v).

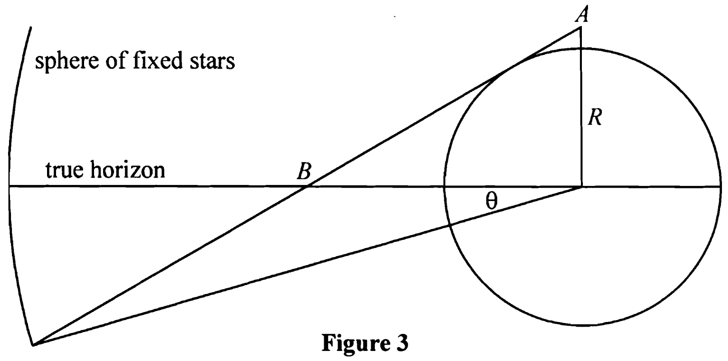


Figure 3

be depressed from the true horizon? And many other [problems] which were detailed [to you] previously.

In my commentary on this problem (see Fig. 3), I have used the celestial dimensions given by al-Kāshī in his *Sullam al-samā'* [al-Kāshī 1881/82: 34–36], where *R*, the radius of the earth, is 1272 *parasangs* (one *parasang* is 12,000 cubits) and the minimum distance of the sphere of the fixed stars is 26,328 *R*, and thus I have calculated *AB* approximately equal to 1477 *R* and θ approximately equal to 2′12″. In the letter known since 1859, al-Kāshī refers to "the higher sphere" (*falak-i aʿlā*), the radius of which is equal to the maximum distance of the fixed stars, and he gives the value 26,340 *R* for it in his *Sullam al-samā'*. Using this value, one obtains the same result, because it is very close to the minimum distance of the sphere of fixed stars.

In the astronomical texts of the Medieval Islamic period three types of horizon are defined (see Fig. 4): 1) the true horizon (*ufuq ḥaqīqī*), i.e., the great circle on the celestial sphere produced by the plane passing through the center of the universe; 2) the sensible horizon (*ufuq ḥissī*), i.e., the intersection of a plane tangent to the earth at the observer's position with the celestial sphere; and 3) the shield-like horizon (*ufuq tursī*, also *ufuq ḥissī* in a more general sense), i.e., the circle drawn on the celestial sphere by the visual rays issuing from the observer's eye tangent to the surface of the earth. Depending on the height of the observer and the place

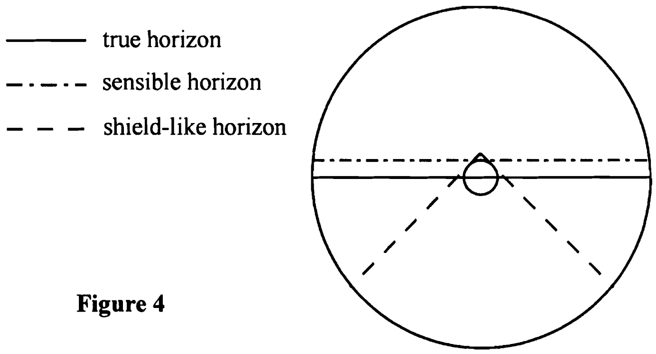


Figure 4

where he stands, this circle may be a small circle or a great circle coinciding with the true horizon. This is the actual boundary between the visible and invisible parts of the sky. The problem in al-Kāshī's letter actually refers to the position of the true horizon relative to the shield-like horizon.

In his commentary on Chaghmīnī's *al-Mulakhkhaṣ fī 'l-hay'a*, Qāḍī-zāda Rūmī defines the shield-like horizon without using the adjective *tursī* [Rūmī 1881/82: 44]. In his Arabic commentary on Ṭūsī's *Tadhkira*, Bīrjandī mentions the three types of horizon using the adjective *tursī* for the third one.

The depression of the visible horizon was discussed by Abū Sahl al-Kūhī (d. about A.D. 1014) in his treatise *Fī ma'rifat mā yurā min al-samā' wa-l-baḥr* (On knowing the visible parts of the sky and the sea). In this treatise al-Kūhī provides the geometric solution to the problem, but he gives no numerical value.[2]

This work of al-Kūhī is mentioned by Ibn al-Haytham (d. after 1040 in Cairo) in a fragment of his treatise *Risāla fī anna al-ẓāhir min al-samā' akthar min niṣfi-hā* (Treatise on [proving] that the apparent [part] of the sky is more than half of it) kept in Oxford Bodleian as Thurston 3 (fol. 116r) and Marsh. 713 (fols. 232r–232v). Sezgin mentions a third manuscript of this fragment kept in Alexandria [Sezgin 1978: 260]. In another fragment of this treatise, kept in Oxford Bodleian, Thurston 3 (fol. 104r), Ibn al-Haytham says that if the height of the observer's eyes is 3.5 cubits, the visible part of the sky will be 4′26″ greater than its invisible part. In fact, if θ is the depression of the visible horizon, then the visible part of a great semicircle from the zenith to the nadir will be $90 + \theta$ degrees and its invisible part will be $90 - \theta$ degrees. Therefore the visible arc is 2θ greater than the invisible arc. Ibn al-Haytham's result is very close to 2θ as calculated by me (4′24″) on the basis of al-Kāshī's parameters.

In his *Kashf 'awār al-munajjimīn* (Uncovering the shortcomings of the astronomers, MS Leiden Or. 98/1, fols. 75v and 84r) Samaw'al b. Yaḥyā (d. about 1174/5) refers to both Ibn al-Haytham's and al-Kūhī's works and quotes from them.

Kamāl al-Dīn al-Fārisī (d. A.D. 1317/18) wrote a commentary entitled *Tanqīḥ al-Manāẓir* (Revision of the *Optics*) on Ibn al-Haytham's *Manāẓir* (*Optics*). In the 7th Book of this work [al-Fārisī 1929, 2: 156], al-Fārisī says that the refraction of light affects our vision of the celestial bodies and the distances between them in two ways:

1) If we apply the correction for refraction, the distance of the moon is found to be less than that given in the astronomical tables, and therefore, the distances and sizes of the other celestial bodies are less than the values given in the tables;
2) Refraction also affects the part of the sky which is visible from a high point. Al-Fārisī adds that Abū Sahl al-Kūhī and others have written treatises about this problem.

2 I have seen the ms. of this work kept in the Holy Shrine Library (Mashhad, Iran) as MS 5412, copied in A.D. 1253/4. For other mss., see [Sezgin 1974: 320].

He also adds that refraction affects the visible part of the sky when the height of the eyes from a level ground or the surface of a sea is three ⟨and a half⟩ cubits. Al-Fārisī says that Ibn al-Haytham calculated the visible part of the sky for this case, without taking refraction into account.

In his *Taḥdīd nihāyāt al-amākin li-taṣḥīḥ masāfāt al-masākin* (Determination of coordinates of positions for the correction of distances between cities), Abū'l-Rayḥān al-Bīrūnī discusses the depression of the visible horizon in connection with the radius of the earth, but not in relation to the radius of the universe [al-Bīrūnī 1967: 183–186].

After my publication of the Persian text and the English translation of al-Kāshī's newly found letter, my colleague Tofigh Heidarzadeh pointed out to me that a short unpublished Arabic treatise in the former Senate library (Tehran), namely, MS 89/2, fols. 14–15, contains a solution to this problem. The treatise is attributed to Mawlānā Afḍal al-Kāshī, but in a marginal remark this authorship is doubted. I will show that this treatise is most probably by Jamshīd al-Kāshī himself, and not by Afḍal al-Kāshī, who is also known as Bābā Afḍal. He was an Iranian mystic poet of the 13th century, who was familiar with astronomy. His full name is Afḍal al-Dīn Muḥammad b. Ḥusayn Kāshānī (or Kāshī). The Arabic text of the short treatise, which consists of three parts, as well as my English translation are reproduced as an appendix to the present paper.

In the first part of the treatise, the numerical solutions are given as follows:

> If a person of 3.5 cubits height stands on a circular place on the surface of the earth, any ray issuing from his eyes, tangent to the surface of the earth extending towards the sky, will intersect the ⟨true⟩ horizon according to Euclid's ⟨parallel⟩ postulate. We ⟨may⟩ extend it towards the higher sphere. Then the distance between the center of the ⟨true⟩ horizon and the ⟨intersection⟩ point of the above-mentioned line of the ray will be approximately 22,566,113,141 cubits; and the above-mentioned line of the ray will join the higher sphere such that the Sine of the arc of the depression will be approximately 191,134,838 cubits.[3]

The methods for finding these values are explained in the second part. Using the notation of Fig. 5 on page 365 and taking $HI = h$, $EA = R$, we obtain the following modern formulas:

$$EI = \frac{R^2}{\sqrt{[R^2 - R^4/(R+h)^2]}}$$

and

$$ZN = R(EZ - EI)/EI.$$

The proofs of the validity of these methods are presented in the third part of the treatise. From the formula for EI we conclude:

$$R \cong \sqrt[3]{2h \cdot EI^2}.$$

3 "Sine" stands for the sine in a base circle with radius R, i.e., $\mathrm{Sin}\, x = R \cdot \sin x$. Cosine and Cotangent used later in this paper are similarly defined by $\mathrm{Cos}\, x = R \cdot \cos x$ and $\mathrm{Cot}\, x = R \cdot \cot x$.

Using the two above-mentioned values $h = 3.5$ and $EI = 22{,}566{,}113{,}141$, we find $R \cong 15{,}275{,}796$ cubits = 1272.93 *parasangs*. In his *Sullam al-samā'*, al-Kāshī gives $R = 1272$ *parasangs*, which is quite close to what we have calculated here; the difference may be due to the approximations which we have made [al-Kāshī 1881/82: 34]. On the other hand, Bābā Afḍal gives the diameter of the earth equal to 2267 *parasangs*, which corresponds to $R = 1133.5$ *parasangs*, differing more than 10 % from the R calculated above. Bābā Afḍal presents his value for the diameter of the earth in an appendix (*mulḥaq*) to his *'Araḍ Nāma* (Book on Accidents) published in his collected works [Bābā Afḍal 1987: 244].

From the formula for ZN, we conclude:

$$EZ \cong ZN\sqrt{(R/2h)}.$$

Substituting the numerical values for ZN (from the treatise published here), R (al-Kāshī's value from *Sullam al-samā'*), and $h = 3.5$ cubits, we find

$$EZ \cong 282{,}244{,}190{,}500 \text{ cubits} = 23{,}520{,}349.21 \textit{ parasangs}.$$

This value differs considerably from the value 26,328 R or 33,509,188 *parasangs*, which is the minimum distance of the sphere of fixed stars according to al-Kāshī in his *Sullam al-samā'*. Since $23{,}520{,}349.21 : 33{,}509{,}188 = 1 : 1.425 \cong 1 : \sqrt{2} = 1 : 1.414$, the difference may be explained as follows: al-Kāshī used an erroneous method of computation, equivalent to the formula

$$EZ \cong ZN\sqrt{(R/h)}.$$

Thus his result could have been 33,262,796.84, which differs by only 0.74 % from 33,509,188. The difference may be due to rounding and approximation errors. If al-Kāshī had not made this mistake, he would have found the Sine of depression as $(191{,}134{,}838) \cdot \sqrt{2} = 272{,}433{,}466.5$. On the other hand, the value $EZ = 23{,}520{,}349.21$ *parasangs*, which we calculated above, is equivalent to $EZ \cong 18{,}490\,R$. But the minimum distance of the sphere of the fixed stars according to [Bābā Afḍal 1987: 245], which can be inferred from his description of the thickness of the celestial spheres, is about 43,800 R or 49,647,300 *parasangs*. Therefore, the treatise presented here is not based on Bābā Afḍal's parameters.

There is more evidence in favor of al-Kāshī's authorship. If the problem had been discussed by Bābā Afḍal, who lived in Kāshān about two centuries before al-Kāshī, it could not have been a new problem for al-Kāshī and his father, and al-Kāshī could have referred to it as Bābā Afḍal's problem. Although Bābā Afḍal was familiar with astronomy, he could not be compared with al-Kāshī or Qāḍī-zāda Rūmī. So it is very likely that because of the similarity in names, the treatise has erroneously been attributed to Afḍal al-Kāshī instead of Jamshīd al-Kāshī. Moreover, we find the term *falak-i a'lā* or *falak al-a'lā* ("the higher sphere") in one of al-Kāshī's letters and also in the short treatise presented here, while Bābā Afḍal does not use this term in his writings.

Later, I found another treatise containing this solution in MS 3585/7 kept in the Malik Library in Tehran. In the table of contents at the beginning of the codex, the title is given as the treatise of Mīrzā Kāfī al-Qā'inī on the difference between the two horizons. Here the proof and description are similar to those in the treatise wrongly attributed to Afḍal al-Kāshī, but no numerical values are given. A marginal note presents an alternative method equivalent to

$$EI = \frac{R(R+h)}{\sqrt{(R+h)^2 - R^2}},$$

which is a simplified form of the above-mentioned formula for *EI*.

The note was added by a certain Muḥammad Mahdī al-Ḥusaynī al-Mūsawī, who says that this solution is simpler than that given by his grandfather's uncle in Quhistān. Quhistān was a historical region in the province of Khurāsān that extended from south Nīshābūr towards south-east Iran up to Sīstān and that included Būzjān in the 10th century A.D.; its capital was Qā'in. This town was a center of scientific activity in the Islamic period, and some mathematicians from Qā'in are known to us.

Another marginal remark states that *ZN* is not really the desired Sine, because the arc *ZK* is not centered at *E*.

Thus the problem of the depression of the visible horizon was proposed by al-Kūhī and Ibn al-Haytham, appeared in Samarkand four centuries later, was transmitted from Samarkand to Kāshān through al-Kāshī's correspondence with his father, and found its way also to Quhistān in eastern Iran.

Acknowledgment

I thank Dr. Charles Burnett from the Warburg Institute (London) and Dr. Hans van de Velde from Leiden University Library for making copies of manuscripts available to me. I am also grateful to Dr. Jan P. Hogendijk for his precious comments on this paper.

Appendix

On the following pages I present an edition of the Arabic text of the short treatise wrongly attributed to Bābā Afḍal al-Kāshī together with an English translation on facing pages. My additions to reconstruct the original text are given in angular brackets ⟨ ⟩. Abundant words of the text are shown in square brackets [ ]. Parentheses are used for my own numbering of the paragraphs.

Arabic text

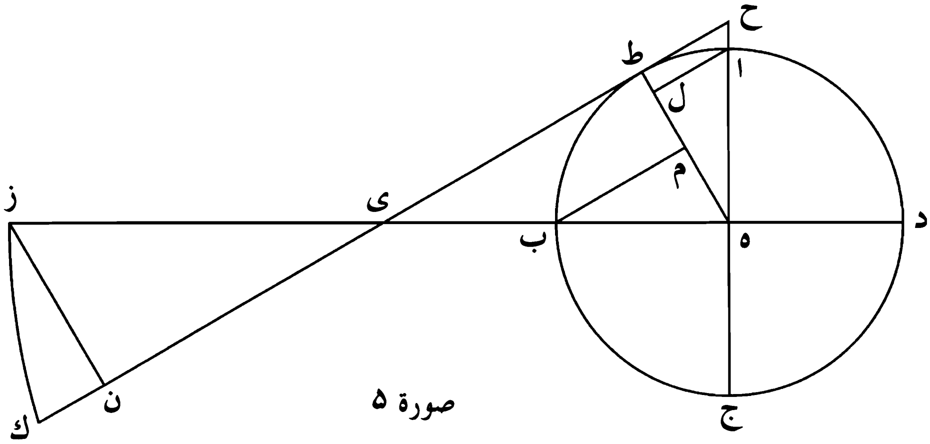
لمولانا أفضل الكاشى رحمه الله تعالى

بسم الله الرحمن الرحيم

(1) اذا قام شخص طول قامته ثلث أذرع و نصف فى موضع الإستدارة على وجه الأرض فالشعاع الخارج من بصره المماسّ لسطح الارض الممتدّ الى جهة السماء يقطع الأفق الحقيقى بحكم مصادرة اقليدس و نتعداه الى الفلك الأعلى فالبعد بين مركز الأفق و موضع الخط الشعاعى المذكور ٢٢٥٦٦١١٣١٤١ ذراعاً تقريباً و يصل[4] الخط الشعاعى المذكور الى الفلك الأعلى على إنحطاط قوس جيبها ١٩١١٣٤٨٣٨ ذراعاً تقريباً

(2) و طريق معرفة ذلك ان نقسم مربع ذراعات نصف قطر الارض على ذراعات نصف القطر مضافاً اليها طول قامة الشخص و نسقط مربع الخارج عن مربع نصف القطر ⟨و نقسم على جذر الباقى مربع نصف القطر⟩ حتى يحصل المقدار الأول فاذا نقصناه من نصف قطر العالم و ضربنا الباقى فى نصف قطر الأرض و قسمناه على المقدار الأول حصل المقدار الثانى

(3) و لنفرض لبيان ذلك **ا ب ج ⟨د⟩** كرة الأرض على مركز **ه** و **ه ز** نصف قطر العالم و **ا ح** طول قامة الشخص و **ح ط ى ك** الخط الشعاعى المماس للأرض على نقطة **ط** القاطع للأفق على **ى** و للفلك الأعلى ⟨على⟩ **ك** ثم نصل بين المركز و نقطة **ط** بخط **ه ط** ⟨و نخرج من **ا** عمود **ال** على **ه ط**⟩ و من **ب** عمود **ب م** عليه و من **ز** عمود **ز ن** على الخط الشعاعى على هذه الصورة



صورة ٥

4 In the text, it is written نصل instead of يصل, which suits the context better.

Translation

By Mawlānā Afḍal al-Kāshī — may God the Almighty have mercy upon him!
In the name of God, the compassionate, the merciful.

(1) If a person of 3.5 cubits height stands on a circular place on the surface of the earth, any ray issuing from his eyes, tangent to the surface of the earth extending towards the sky, will intersect the ⟨true⟩ horizon according to Euclid's ⟨parallel⟩ postulate.[5] We ⟨may⟩ extend it towards the higher sphere. Then the distance between the center of the ⟨true⟩ horizon and the ⟨intersection⟩ point of the above-mentioned line of the ray will be approximately 22,566,113,141 cubits, and the above-mentioned line of the ray will join the higher sphere such that the Sine of the arc of the depression will be approximately 191,134,838 cubits.

(2) The way to find this ⟨is as follows:⟩ We divide the square of the radius of the earth in cubits by the radius of the earth plus the height of that person in cubits. Then we subtract the square of the result from the square of the radius ⟨and we divide the square of the radius by the square root of the remainder⟩, so that the first magnitude will be obtained. If we subtract it from the radius of the universe and multiply the remainder by the radius of the earth and divide it by the first magnitude, the second magnitude will be obtained.

(3) To explain this, we assume *ABGD* as the sphere of the earth with the center *E*, *EZ* as the radius of the universe, *AH* as the height of the person, and *HTIK* as the ray tangent to the earth at the point *T*, intersecting the horizon at *I* and the higher sphere at *K*. Then we join the center and the point *T* by the line *ET* ⟨and from *A* we draw *AL* perpendicular to *ET*⟩, from *B*, *BM*, perpendicular to it, and from *Z*, *ZN* perpendicular to the line of the ray, as shown in this figure:

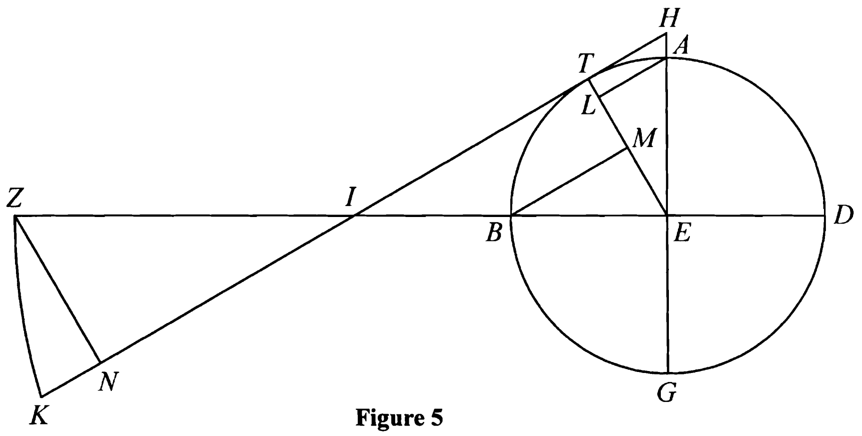


Figure 5

5 Postulate 5 in the *Elements* [Euclid 1956, I: 155].

فمثلثا ا ل ه ح ط ه متشابهان لإشتراك زاوية ه وكون ط و ل قائمتين فنسبة ح ه الى ا ه أعنى ه ط كنسبة ه ط الىه ل بالرابع من سادسة الأصول فاذا ضربنا نصف القطر أعنى ه ط فى نفسه و قسمناه على ح ه يكون الخارج مقدار ه ل لما تقرر فى الأصول أن مسطح الوسطين فى الأربعة المتناسبة إذا قسم على أحد الطرفين يكون الخارج الطرف الآخر ثم إن ه ل جيباً لقوس ب ط لما تقرر أن ⟨جيب⟩ تمام اى قوس الى نصف القطر جيباً لتمام تلك القوس فيكون مساوياً لخط ب م و اذا نقصنا مربع ب م عن مربع ب ه و اخذنا جذرالباقى حصل مقدار ه م بإستبانة شكل العروس فنقول مثلثا ه ب م ه ى ط ايضاً متشابهان متساويا الزوايا كل لنظيرها و نسبة ه م الى ه ط أعنى ه ب كنسبة ه ب الى ه ى فنضرب ب ه فى نفسه و نقسم على م ه حتى يحصل ه ى و هو المقدار الأول

(4) ثم إذا نقصنا⟨ه⟩ عن ه ز نصف قطر العالم بقى ز ى معلوماً فمثلثا ز ن ى ه ط ى متشابهان متناسبان بمثل ما ذكرنا أعنى نسبة ه ط الى ن ز كنسبة ه ى الى ز ى و المجهول هيهنا أحد الوسطين فنضرب ز ى فى ه ط و نقسم الحاصل على ه ى فالخارج من القسمة ز ن و هو المقدار الثانى

(5) و بطريق آخر نقول بعد استخراج قوسى ا ط ب ط كما ذكر نقسم جيب احديهما على [تمام] جيب الأخرى ليحصل ظل كل واحد من القوسين أعنى الظل المصطلح لها و هو ح ط ى ط ثم نتمم التقريب بمثل ما ذكرناه و لو ظهر تفاوت بين ما إستخرجناه و ما إستخرجه غيرنا فلعله لإهمال بعض الكسور في أحد الحسابين و الأمر فيه سهل إذا المقصود فى أمثال هذه التعريف لا التحقيق و التدقيق و الله ولى التوفيق تم

Then, the triangles *ALE* and *HTE* are similar because they have the angle *E* in common, and *T* and *L* are right angles. Then the ratio of *HE* to *AE*, that is *ET*, is equal to the ratio of *ET* to *EL* according to *Elements* VI.4. If we multiply the radius, i.e. *ET*, by itself and divide it by *HE*, the result will be the magnitude of *EL*, because it is established in the *Elements* that for any four proportionals, if we divide the product of the means by one of the extremes, the other ⟨extreme⟩ will result.[6] Then, *EL* is the Sine of the arc *BT*, because it is established that the Cosine of any arc ⟨reaching⟩ up to the radius, is the Sine of the complement of that arc, and thus it is equal to the line ⟨segment⟩ *BM*. If we subtract the square of *BM* from the square of *BE* and take the square root of the remainder, the magnitude of *EM* will result according to the Pythagorean theorem.[7] Then we say that the triangles *EBM* and *EIT* are also similar and their corresponding angles are equal. The ratio of *EM* to *ET*, that is *EB* is equal to the ratio of *EB* to *EI*. We multiply *BE* by itself and divide by *ME* so that *EI* results, and it is the first magnitude.

(4) Then, if we subtract it from *EZ*, the radius of the universe, the remainder *ZI* will be known. The triangles *ZNI* and *ETI* are similar and proportional as we have said, i.e., the ratio of *ET* to *NZ* is equal to the ratio of *EI* to *ZI*, and here the unknown is one of the means. We multiply *ZI* by *ET* and divide the result by *EI*, the quotient is *ZN* and this is the second magnitude.

(5) In another way, we say: After finding the arcs *AT* and *BT* as mentioned, we divide the Sine of one of them by the Sine of the other to find the Cotangent of any of the two arcs, i.e., the Cotangent conventionally used for it: *HT* and *IT*. Then we complete the approximation as we have explained. Should there appear any discrepancy between what we found and what someone else found, it may be due to neglecting some fractions in one of the calculations. This is an easy matter because in such cases a ⟨general⟩ characterization is intended, not an accurate and precise ⟨numerical⟩ result. God provides success. The end.

6 See *Elements* VII.18 [Euclid 1956, II: 318–319].

7 Proposition of the Bride given in *Elements* I.47 [Euclid 1956, I: 349–350].

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