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KŪSHYĀR IBN LABBĀN'S MATHEMATICAL APPROACH IN HIS ASTRONOMICAL HANDBOOK

Kūshyār ibn Labbān ibn Bāshahrī al-Jīlī also named Kūshyār Gīlānī was born in Gīlān province on the southern coast of the Caspian Sea in 963 AD. So in this Iranian year which ends in 20 March 2014 several gathering are held in Iran for his 1050th birth anniversary. In the University of Gilan situated in Rasht, the center of Gīlān province, an archive for Kūshyār's scientific legacy is established in which copies of the manuscripts of his works from all over the world and papers and books related to him (editions, translations, researches, etc.) are collected. Several scientific institutions are named after him in Gīlān.

Kūshyār left Gīlān for Rayy (near present Tehran) where he met Abū Rayhān Bīrūnī and Abū Maḥmūd Khujandī and exchanged scientific knowledge with them. He later moved to old Gorgān in Ṭabaristān province on the southern coast of the Caspian Sea, adjacent to Gīlān. Old Gorgān was ruined in the Mongols' rush in the 13th century AD. Nowadays there is a city named Gorgān around 80 Kms. away from old Gorgān whose ruins are near the city Gonbad-e Kāvūs. Kūshyār wrote his famous astronomical treatise *al-Zīj al-Jāmi*ʿ in old Gorgān under the patronage of the local ruler Kāvūs from Ziyarid dynasty (10th and 11th centuries AD). From two partial manuscripts of this *zīj* preserved in Cairo, we have recently known that Kūshyār observed a conjuction of Mars and Saturn in July 993. Based on a reference by Kūshyār's disciple ʿAlī ibn Aḥmad Nasawī who wrote a commentary of Kūshyār's *zīj*, Kūshyār died before 1047.

Five works by Kūshyār, all in Arabic, the lingua franca of his time, are known to us, one on arithmetic, two *zījes*, his astrological treatise and his book on astrolabe. Most of these works have been edited

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and old or modern translations of them are published with introductions and commentaries.

Kūshyār's astrological work *al-Madkhal ilā 'ilm aḥkām al-nujūm* (Introduction to Astrology) has been published in 1987 with an English translation (by Michio Yano) to which an old Chinese translation his also appended. In the introduction of this work Kūshyār mentions his two astronomical books *al-Zīj al-Jāmi'* (The comprehensive astronomical handbook) and *al-Zīj al-Bāligh* (The eloquent astronomical handbook). The latter has not remained in complete form, and only a short chapter of it exists in a manuscript in Bombay. The former, *al-Zīj al-Jāmi'* is preserved in three complete and several incomplete manuscripts. There are also several manuscripts of it written in Hebrew alphabet which cover the whole text altogether.

Kūshyār composed his *al-Zīj al-Jāmi'* between 1020 and 1025 AD in the theoretical tradition of Ptolemy and based on the observations made by al-Baṭṭānī. In the first chapter of the fourth section of Book I of the *zīj*, he says that he studied the ancient astronomical observations and the recent ones and checked them through conjunctions and meridional altitudes, for many years. He found al-Baṭṭānī's observations most correct with least defects and inconsistencies, and also closest to is time. He also admires al-Baṭṭānī's relying on Ptolemy's *Almagest*. In Kūshyār's short note about his observation of the conjunction of Mars and Saturn, he says:

> Then I found their true longitudes for the <local> meridian, in order to make it sure for myself that one may not rely on observational *Zīj*es, because if they turn out to be correct in one <case>, by some tricks, they are erroneous in ten <cases>. There is no way to correcting them by the scientists of our time and by efforts of our rulers.

Kūshyār says that he adjusted the observed values of al-Baṭṭānī from Raqqa (in Mesopothemia) into a locality with geographical longitude 90° from the Canary Islands for which the difference in local time compared to Raqqa is one hour and 7 minutes. According to Kūshyār's table for the geographical coordinates of localities, the longitude of [old] Gorgān is 90°.

Kūshyār's *zīj* consists of four *maqālas* or four books. According to Kūshyār, Books I and II on elementary calculations and numerical tables constitute the practical part of the treatise. The next two books are its theoretical part: Book III on cosmology (*hay'at*) which presents a description of the configuration of the Universe mainly based on Ptolemy's astronomical models; Book IV on proofs (*barāhīn*) contains demonstrations of the validity of the calculation methods provided in Book I.

In 1090 AD (around 70 years after its composition), Muḥammad ibn 'Umar ibn Abī Ṭalib Munajjim Tabrīzī translated Book I of Kūshyār's *zīj* into Persian. A unique manuscript of this translation is extant in the library of Leiden State University. I have prepared an edition of the part on calendars which is published in *Tarikh-e Elm* (Iranian Journal for the History of Science) in 2008.

Ludwig Ideler published an edition of some fragments on calendars with German translation in 1825–1826. Joachim Lelewel compared some data about geographical coordinates in this work and those of Bīrūnī and Ibn Yūnus in 1852. Eilhard Wiedeman translated the preface of Kūshyār's *zīj* into German in 1920. An edition of a chapter from Book III on the distances and sizes of the celestial bodies was published in Hyderabad (India) in 1948. A new edition of this chapter with English translation and commentary is recently published in Germany by Jan. P. Hogendijk, M. Yano and me (*Zeitschrift für Geschichte der arabischen-islamischen Wissenschaften*, 19 [2010–2011], Frankfurt).

Prof. Kennedy has given a summary of the contents of the *zīj* in his famous *Survey of the Islamic Astronomical Tables* in 1956, and an account of Kūshyār's method for calculation of equation of time in a paper published in 1988. Dr. Benno van Dalen analyzed the table for equation of time in Book II, in 1994. Prof. J. L. Berggren published a translation of Section 3 of Book IV on spherical trigonometry with his commentary in 1987.

My edition and English translation of Books I and IV with commentary has been published in Frankfurt in 2009. These two Books are of highly mathematical nature.

In his *al-Zīj al-Jāmi'*, Kūshyār generally uses a mixed sexagesimal numeration system both for integers and fractions. The sexagesimal digits (from 1 to 59) are shown by the alphanumerical *abjad* system. This numeration method was called *ḥisāb al-munajjimīn* (numeration used by astronomers). However, in some cases he also uses Hindu-Arabic numerals or describes the numbers by words. Kūshyār uses

Persian conventional years because they have not leap years and the number of the days in a year is an integer (365) and the number of the days in each month is the same (30). He gives his formulas for astronomical computations by words in the 85 chapters of Book I. His proofs of these formulas in Book IV are accompanied by figures.

Book I starts with a section on calendars, in which Kūshyār describes the different systems of the Greek, Arabic and Persian calendars. He discusses the methods for defining leap years in each system, the way to find the week-day of any date in these calendars, and the methods for converting a date in one of these calendars into the corresponding date in another one. Here he uses simple arithmetic calculations in which he finds the residue of the number of days in a certain period, with modulus 7 (congruence calculation). He also mentions how to use the corresponding tables in Book II for such purposes.

The next section deals with Sines and Chords. Kūshyār follows Ptolemy in taking the radius of the circle equal to 60 parts. So his Sine and Cosine functions are 60 times greater than the modern ones (here we show them with capital initial letters). This has the consequence that Kūshyār often has to divide the product of any two of these functions by 60. He calls this division «lowering» (*inhiṭāṭ*) which is very simple in the sexagesimal numeration system: you just shift the sexagesimal point one order to the left.

Kūshyār gives an approximate value of Sine of one degree equal to 1;2,49,38,31 which corresponds to 0.017452046×60 in decimal system. Its modern value is 0.017452406 which equals to 1;2,49,43,11/60. Using this value of Sine of one degree, he calculated the Cosine of one degree equal to 59;59,27,6,12,39 which corresponds to 0.999847701×60 in decimal system. Its modern value can be found as 0.999847695 which equals to 59;59,27,6,7,45/60 shaded digits show the correct ones in Kūshyār's calculation.

Kūshyār also describes the following trigonometrical identities which are used in constructing a Sine table:

$$\mathrm{Sin}^2 x + \mathrm{Cos}^2 x = 60^2$$

$$\mathrm{Sin}(x+1) = \mathrm{Sin}x\ \mathrm{Cos}1°/60 + \mathrm{Cos}x\ \mathrm{Sin}1°/60$$

Kūshyār gives a linear interpolation method for finding the Sines when the argument is not an integer number of degrees. He also introduces the following formulas to be used in checking the correctness of a result for Sin$x$:

$$\mathrm{Sin}\,(2x) = 2\mathrm{Sin}x\mathrm{Cos}x/60$$
$$\mathrm{Sin}(x+y) = [\mathrm{Sin}x\mathrm{Cos}y+\mathrm{Cos}x\mathrm{Sin}y]/60$$

In Book II, a Sine table provides the Sine values and the differences of consequent tabular values for 1, 2, …, 90 degrees, accurate to seconds. In some manuscripts (Berlin and Leiden) the Sine table is much more detailed and besides the integer numbers of degrees, for each degree plus 1, 2, …, 15 minutes and also plus 18, 21, 24, …, 57 minutes, the Sine is given accurate to fourths.

Here he mentions the function Sagitta (*sahm*) such that in modern notation it is equal to 60–Cos$x$. Other mathematicians including Bīrūnī used the name *al-jayb al-ma'kūs* (Versed Sine) for this function and they used the term sagitta for the function 60–Cos(x/2). Kūshyār's application is rather strange. But in his *Glossary of Astronomy*, an independent chapter at the end of Book III, he is consistent with Bīrūnī and others.

In the Sagitta table provided in Book II, the tabular values for 1, 2, …, 90 degrees are given accurate to seconds and calculated in accordance with Kūshyār's unusual terminology.

Ptolemy used the Chord as a basic trigonometrical function. Kūshyār has allocated some chapters to the Chords, although he once says that we do not need the Chords in this treatise. He formulates the relation between Sines and Chords as:

$$\mathrm{Chord}x = 2\mathrm{Sin}(x/2).$$

In Book IV (on proofs), he actually uses Chords and provides methods for calculation of chords of supplement of an arc, half an arc, the sum of two arcs, and the difference between two arcs. He also calculates the chords of arcs of 90, 120, 36, and 72 degrees. Finally he presents a calculation method for the Chord of one degree.

In Section 3, Kūshyār gives the definitions of Tangent and Cotangent functions. The term Tangent (*ẓill*) literally means «shadow» in Arabic. More specifically, he calls it «first shadow» or «reversed shadow». The terms for cotangent (*ẓill thānī, ẓill mustawī*) literally mean «second shadow» and «horizontal shadow». He also mentions Secant and Cosecant as hypotenuse of the first shadow (*quṭr ẓill awwal*) and hypotenuse of the second shadow (*quṭr ẓill thānī*).

For Tangents, he takes the length of the gnomon equal to 60 parts. For Cotangents he takes it equal to 7 or 12 parts, where the units for shadows are feet or digits, respectively.

He also mentions the following relations between these trigonometrical functions:

(i) $\mathrm{Tan}h = \mathrm{Cot}(90° - h)$
(ii) $a\mathrm{Tan}h = a/\mathrm{Cot}h$
(iii) $\mathrm{Tan}h = \mathrm{Sin}h/(\mathrm{Cos}h/\mathrm{R})$
(iv) $\mathrm{Sec}h = \mathrm{Tan}h/(\mathrm{Sin}h/\mathrm{R})$
(v) $\mathrm{Sec}h = (\mathrm{Tan}^2 h + \mathrm{R}^2)^{1/2}$
(vi) $\mathrm{Sin}h = \mathrm{Tan}h/(\mathrm{Sec}h/\mathrm{R})$
(vii) $\mathrm{Cot}h = \mathrm{Cos}h/(\mathrm{Sin}h/\mathrm{R})$
(viii) $\mathrm{Cosec}h = \mathrm{Cot}h/(\mathrm{Cos}h/\mathrm{R})$
(ix) $\mathrm{Cosec}h = (\mathrm{Cot}^2 h + \mathrm{R}^2)^{1/2}$
(x) $\mathrm{Cos}h = \mathrm{Cot}h/(\mathrm{Cosec}h/\mathrm{R})$

Bīrūnī in his treatise on *Shadows* refers to this chapter and says:

> What Kūshyār proposes for dividing the Cosine of the altitude by the Sine of altitude, lowered, is exactly what al-Nayrīzī does.... And Abu'l-Wafā' proceeded like him, except that he did not lower it, for he assumed [the length of] the gnomon to be one [unit] ....

Kūshyār presents the values of Tangent for 1, 2, ..., 45 degrees and the consecutive differences accurate to seconds and for R=60. He gives the Cotangents for 1, 2, ..., 90 degrees accurate to minutes.

The remaining five sections of Book I contain astronomical calculation methods in which the mathematical materials provided in Sections 2 and 3 are applied.

In Book IV which is allocated to the proofs of the validity of the calculation methods provided in Book I, the first three sections contain merely mathematical materials. Section 1 deals with finding the Chord of the supplement of an arc, and the Chord of a quarter, a third, a fifth and a tenth of a circle. Kūshyār follows with proving the formulas relating to the Chord of the sum and difference of two arcs. Section 2 discusses Tangents and Cotangents in detail. Section 3 contains four premises on the properties of spherical triangles and three chapters on the properties of proportions. This section was first translated into English by Prof. J. L. Berggren (1987). My translation based on more Arabic manuscripts differs from him in a few cases.

The first premise of Section 3 is a proof of the sine theorem for right spherical triangles. Bīrūnī says that Kūshyār took this theorem from Abū Maḥmūd Khujandī, named it *al-mughnī* (making [one] able to dispense [with Menelaus' theorem]), and abridged Khujandī's proof of it. He adds that Kūshyār did not find the generalized form of the sine theorem. However, this generalized form is found as the third premise in this section. Kūshyār's proof of the generalized form is similar to that of Bīrūnī.

The second premise is a proof of the cosine theorem for right spherical triangles. According to Bīrūnī, Kūshyār's proof is more simple than the proof by Abu'l-'Abbās Nayrīzī and Abū Ja'far Khāzin.

The fourth premise is a proof of the tangent theorem for right spherical triangles which according to Bīrūnī was discovered by Abu'l-Wafā' Būzjānī. Kūshyār's proof is similar to that of Būzjānī.

The remaining sections of Book IV are again of astronomical nature in which the mathematical theorems are applied.

Before finishing my account of Kūshyār's mathematical approach in his *zīj*, I should mention two major innovations by Kūshyār. Using a method called *displacement and shift* by Benno van Dalen, Kūshyār avoids the possible confusion regarding adding or subtracting equations in finding the solar and planetary longitudes. Kūshyār also deviates from Ptolemy by using a different interpolation scheme for finding the anomaly of planets which reduces the number of tables from four to three, although this leads to a less accurate result. Glen van Brummelen who has worked on this achievement mentions that Kūshyār was the first one who presented an alternative to Ptolemaic interpolation for the planetary functions. He adds that while Kūshyār's scheme may be regarded as ingenious, it must also be considered only a partial success.

As conclusion, I may say that Kūshyār's adherence to mathematical firmness of the astronomical calculations and his contribution in modifying the established methods reflects the fact that he may be regarded equally a mathematician and a theoretical astronomer.

His *al-Zīj al-Jāmi'* is one of the most important astronomical works of the Islamic civilization which later influenced the development of astronomy in Europe through North Africa and Southern Spain (al-Andalus).

ABSTRACT

Kūshyār ibn Labbān's Mathematical Approach in His Astronomical Handbook

Kūshyār ibn Labbān composed his famous astronomical work *al-Zīj al-Jāmiʿ* from 1020 to 1025 AD. This work was prepared in the theoretical tradition of Ptolemy and based on the observations of al-Baṭṭānī. Kūshyār composed his treatise in four books: I) Elementary calculations; II) Tables; III) Cosmology; and IV) Proofs. In Book I, consisting of 85 chapters, he presents different calculation methods for astronomical purposes. He provides the proofs of validity of these calculation methods in Book IV. These two books are of highly mathematical nature and Kūshyār applies a firm mathematical basis in them and widely uses spherical trigonometry. In this paper, I will discuss his mathematical approach and I will show that his deviations from Ptolemy reflect his mathematical ability that led to innovations. Kūshyār reports his observation of a conjunction of Mars and Saturn in July 993. He also wrote a treatise on Hindu reckoning which is one of the earliest existing Arabic texts on this subject. From his other astronomical work *al-Zīj al-Bāligh*, only a short chapter has remained. Kūshyār's treatises on astrology and astrolabe are also studied and published.

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