Vol. 21, Nos.1-4 (1999), 1-9

GANITA BHĀRATĪ
BULL. IND. SOC. HIST. MATH.

RECREATIONAL PROBLEMS FROM ḤĀSIB ṬABARĪ'S MIFTĀḤ AL-MU'ĀMALĀT

MOHAMMAD BAGHERI
P.O. Box 13145-1785, Tehran, IRAN
E-mail: bagheri@math.sharif.ac.ir
Fax : +98-21-65 92 01

Recreational mathematical problems have always been used to interest non-specialists in mathematics and to introduce them to its beauty and charm. They are usually mathematical problems clothed in elements from everyday life though not caring for realistic circumstances — so that even non-mathematicians may have enough motivation to meet the challenge of solving those problems.

This point is of great importance for those who seek means of arousing the students' interest in their mathematical lessons. For mathematicians the attraction already exists, and one may say that unsolved problems have played the same role for mathematicians and have pushed them to explore new areas. The underlying mathematical concepts of recreational problems have been spread far and wide by merchants, soldiers, pilgrims, etc., as brain teasers.

The clothing of mathematical problems to make them meaningful for non-mathematicians, bears the imprint of their national and historico-geographical origin. The search for corresponding problems in different cultures can lead to evidences of probable contacts between them. Sometimes the elements constituting the outward clothing of the problems, when transferred to new environments, were adapted to elements peculiar to the new ambience. I will mention some examples later. However, the structural similarity can be helpful in such cases. Another point which shows the degree of probable correspondence between similar problems in different traditions is the equality of the corresponding numerical values; cf., for example, the problems relating to 100 birds in Chinese, Indian, Iranian, Arabic and European works.

I am going to give some interesting examples of recreational problems from an early Persian arithmetic treatise entitled *Miftāḥ al-mu'āmalāt* ("Key to transactions") by Muḥammad bin Ayyūb Ṭabarī, an Iranian mathematician and astronomer from the Iranian province of Ṭabaristān (now called Māzandarān) on the southern coast of the Caspian Sea. He flourished in the last decades of the 11th century C.E. He had written first another Persian work on arithmetic entitled *Shomār Nāme* ("Treatise on calculation"), and later he wrote *Miftāḥ* probably as a complement to it. In a lecture given in the First International Symposium for the History of Arabic Science held in Aleppo, April 1976, the late Prof. Heinrich

Hermelink from Munich referred to this work as "the richest mine" of recreational mathematics in Islamic mathematical tradition (Hermelink, p.46). An edition of *Miftāḥ* by Dr. Moḥammad-Amīn Rīyāḥī has been published in Tehran (1970) from its unique manuscript in Aya Sofya (Istanbul). A brief description of this work is given in Rebstock, pp. 177, 182-192. Ṭabarī has another work on the subject, entitled *al-Mūnis fī nuzhat ahl al-majlis,* a manuscript of which is kept in Rampur (India) (Storey, p.4).

Miftāḥ al-muʿāmalāt, written for non-mathematicians, is one of the most important Persian works on the subject, especially for the Persian màthematical terminology. It consists of 6 chapters, each divided into several short sections. Chapter 4, entitled "On rarities and hidden numbers," contains 54 sections each presenting a problem. Not all of them really belong to recreational mathematics, and I will mention here the interesting cases. Some of these problems are of algebraic nature. However, Ṭabarī gives their solution in arithmetic terms, without reasoning.

Problem 17 in *Miftāḥ al-Muʿāmalāt* is about finding a secret/covert number. Problems of this kind, known as Mużmarāt (thought-of [number]s), are methods of finding out a number someone has thought of, through indirect data we receive from him. In Ṭabarī's *al-Mūnis fī nuzhat ahl al-majlis* the 2nd of the 4 chapters is devoted to the subject of finding secret numbers. Here, in problem 17, we ask someone to choose three integers whose sum is 18. Ṭabarī refers to these unknown numbers as the numbers kept in one's heart, right hand, and left hand. We ask him to multiply the number in his heart by 2, that in his right hand by 17, and that in his left hand by 18, and then tell us the sum. In order to find the three numbers, we subtract this total sum from the square of 18 and divide the remainder by 16. The integer-quotient is the number kept in the hart, and the remainder of division is the number kept in the right hand. Then we can find the number in the left hand by subtracting these two from 18. The underlying algebraic structure is as follows :

$$x + y + z = 18$$
$$18x + 18y + 18y = 324$$
$$2x + 17y + 18z = 254 \text{ (for example)}$$

Subtraction results in $16x + y = 70$, and by division $x + y/16 = 70/16$. Then we have $x = [70/16]$ and $y = 70 — 16x$. Therefore, $x = 4$, $y = 6$ and $z = 8$.

Problem 18 is about a tree 1/3 of whose height is in water, 1/4 in mud and 10 cubits (*gaz*) in the air. The height of the tree is easily found by ordinary arithmetic operations on fractions. An extended version of this problem is given in section 38 where of the height of the tree 1/3 is in water, 1/4 in mud, 1/5 in sand, 1/6 in dry soil, and again 10 cubits in the air. The same elements appear in problem 31 with a different structure, where of the height of the tree 1/2 is in water, 1/3 in mud, and its square root in the air. Here we have :

$x = x/2 + x/3 + \sqrt{x}$, $x/6 = \sqrt{x}$, $\sqrt{x} = 6, x = 36$.

Problem 37 has the same structure as problem 18 but it is presented in terms of a fish whose head and tail constitute 1/3 and 1/5 of its weight respectively, and its body is 10 *mans*.

This problem probably originated in India. Mahāvīra, an Indian mathematician of the 9th century, in his *Gaṇita-Sāra-Saṅgraha* gives a similar problem for a pillar of which 1/8 is in the earth, 1/3 in water, 1/4 in mud, and 7 cubits in the air. In Islamic tradition, this problem was popular in terms of the height of atree, the weight of a fish, the length of a twig or a piece of cloth. Bahā' al-Dīn 'Āmilī has combined two versions asking for the length of a fish 1/3 of which is in the mud, 1/4 in water and 3 spans in air (problem 15, p. 137). The version given in problem 31 by Ṭabarī which involves a square root and leads to a quadratic equation was also inspired by Indian sources; it is found in Mahāvīra and in *Pāṭigaṇita* of Śrīdhara (Hermelink, p. 48).

Ghiyāth al-Dīn Jamshīd al-Kāshī gives another version of this problem as problem 25 in the 4th chapter of the 5th book of his *Miftāḥ al-ḥisāb* (p. 253). His version is as follows: Of a fish the head is 4/9 of the weight, the tail is 5 times the 5th root of the weight, and the rest (i.e., its body) is 8 times the tail. This leads to an equation of the 5th degree which can be easily solved :

$$X^5 = (4/9)X^5 + 5X + 40X,\ (5/9)X^5 = 45X,\ X^4 = 81, X = 3$$

$$X^5 = 243,\ (4/9)X^5 = 108,\ 5X = 15,\ 40X = 120$$

According to Hermelink (p. 48), in the West this problem was called "The rod in the water", and began with Fibonacci (13th century C.E.).

Problem 24 is about finding out who has hidden a ring. Since the supposed keepers stand in a row, the problem is actually reduced to finding an unknown integer n from implicit data. We ask someone who knows n to multiply it by 3/2. We ask him to complete it to the next integer if a half results, and if so, we bend one finger. Then we ask him to multiply the last result by 3/2, and if a half results, to complete it to the next integer, and in this case we bend two fingers. Then we aks him to cast out nines, and for each nine, we bend 4 fingers or add four for each nine. The number of bent fingers equals n. The underlying structure of this interesting mathematical tour de force is as follows :

If $n = 4k$, $6k$ (no bent finger), $9k$ (no bent finger), **k**, $4k$

If $n = 4k + 1$, $6k+2$ (**1** bent fingre), $9k+3$ (no bent finger), **k**, $4k+1$

If $n = 4k+2$, $6k+3$ (no bent finger), $9k+5$ (**2** bent fingers), **k**, $4k+2$

If $n = 4k+3$ $6k+5$ (1 bent finger), $9k+8$ (2 bent fingers), **k**, $4k+3$

This problem was already given in the Arabic arithmetic of Abū Manṣūr ʿAbd al-Qāhir b. Ṭāhir al-Baghdādī (d. about 1037-8), entitled *al-Takmila fi 'l-ḥisāb* (MS 6911, Tehran University, dated 660/1261-2, fol. 104v; al-Baghdādī, p. 291-2). Abū Manṣūr also mentions the remainders of casting out nines in each case, i.e., 3,5 and 8. In West, this problem is given in the works of Beda Venerabilis, Fibonacci, etc. (Tropfke, pp. 643-4).

Problem 27 is about a man who has some silver money. He says that if he also had 1/2 this amount and 1/3 this amount and 1/6 this amount and 5 more **dirhams**, then he would have 20 **dirhams.** This leads to an equation of the first degree: $X + X/2 + X/3 + X/6 + 5 = 20$, giving $X = 7.5$. The problem itself is not a remarkable one. However, the same structure is found in problem 47 in which we find the number n of the pigeons settled on a roof, knowing that the number n, plus n, plus $n/2$, plus $n/4$, plus 1, equals 100. This enigma has been cast into a Persian quatrain which was quite popular in Iran, and most ordinary people have also heard of it. Supposedly, a wild pigeon flying over the roof of a house sees a group of domestic pigeons and asks about their number. The answer is given in the form of an enigmatic poem which may be rendered thus :

Do not taunt us with being small in number,
Because we are domestic and we multiply infinitely;
We plus we plus half of us and quarter of us,
If you join us, we become one hundred.

The answer is found by the linear equation $X + X + X/2 + X/4 + 1 = 100$, which yields $11X/4 = 99$, and $X = 36$.

In problem 42, related to geometry, of a tree 3 cubits was outside water level. A wind blew, and bent it so that its top reached the water level, 9 cubits apart. How tall is the tree (Fig. 1)?

Ṭabarī gives the solution in an algorithmic form, as usual, without any reasoning (based on theorem 34 of the 3rd book of *Elements*):

9x9 = 81

81:3 = 27

27+3 = 30

30:2 = 15

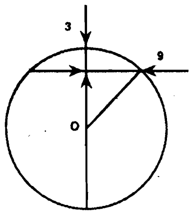

Fig. 1

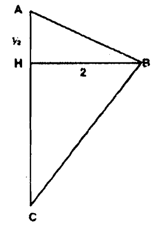

Fig. 2

This problem is found in ancient Chinese arithmetic texts. It is mentioned as problem 6 in book 9 of *Jiu jang suan shu* ("Nine chapters on the art of mathematics"), and as problem 13 in book 1 of *Zhang Qiu Jian suan jing* ("Zhang Qiu-Jian's mathematical manual"), where the same story about the bending of a reed is presented (Qian Baocong, Vol. 1, p. 243; Vol. 2, p.338)[1]. The same geometrical structure is found in another problem given in the "Nine chapters" where a cylindrical wood log, partly embedded in a wall, is sawed to the depth of one *ku*, the width of the sawed section is 10 *kus*, and the radius of the wood log is required.

In Indian mathematics, the problem is found in Bhāskara II's *Līlāvatī* (no. 153, p. 66) in the following poetic words: "In a certain lake swarming with ruddy geese and cranes, the tip of a lotus bud was seen a span above the surface of water. Forced by the wind, it gradually moved, and was submerged at the distance of two cubits. Compute quickly, o mathematician, the depth of water." He solves it by an algebraic method (Fig. 2):[2]

$$BC - HC = AC - HC = 1/2$$

$$BC + HC = (BC^2 - HC^2) / (BC - HC) = HB^2 : 1/2 = 4 : 1/2 = 8$$

Subtracting the corresponding sides results in the requested depth :

$HC = 3.75$ cubits.

Al-Kāshī also gives a version of this problem in his *Miftāḥ al-ḥisāb*, as the first, of 8 geometrical problems (p. 263). "A spear is planted in water with 3 cubits of its height outside water. A wind inclined it so that it submerged in water and its tip reached the surface of water 5 cubits away. How long is the spear?" (Fig. 3)

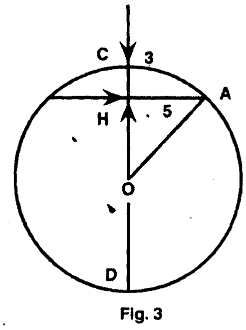

Fig. 3

By the same method as Ṭabarī, al-Kāshī says that square of AH divided by HC gives HD: $5 \times 5 : 3 = 25/3$. Adding HC gives $CD = 34/3$. This is the diameter, and by halving it the radius is found, which equals the length of the spear: $34/3 : 2 = 17/3 = 5+2/3$ cubits.

Al-Kāshī also solves it by an algebraic method as follows:

$$HO = X, X^2 + 25 = AO^2 = OC^2 = (X+3)^2$$

$$X^2 + 25 = X^2 + 6X + 9, \quad 6X = 16 \quad X = 8/3$$

$$8/3 + 3 = 17/3 = 5 + 2/3.$$

This problem is also found in a noval entitled *Kavanagh* by the American poet Henry Wordsworth Longfellow (1807-82), who was also interested in mathematics and believed that recreational problems would appeal to the fancy of the student instead of following the dry, technical language of the textbooks. Sam

1. I thank Prof. Joseph W. Dauben for kindly sending the information on the Chinese sources to me.
2. Indian problem of this type is found much earlier in Bhāskara I's commentary (A.D.629) on the *Āryabhaṭīya*; see *Gaṇita Bhāratī*, vol. 11 (1989), p.43. — Editor (RCG)

Loyd (1841-1911), the famous American puzzlist, writes that Longfellow discussed this puzzle with him as the "Lily problem" about a waterlily growing in a lake. Loyd gives the same numbers as Bhāskara II, but his solution is like Ṭabarī, based on the property of intersecting chords in a circle (Gardner, p.23).

Problem 43 is about 10 weighing stones with which all weights from 1 to 10000 **dirhams** and even more can be weighed. One **dirham** is about 15 grams. This is a famous and popular problem. Ṭabarī gives the weights as equal to 1, 3, 9 and so on, to the 9th power of 3, which is 19683. Ibn al- Bannā (d. 1321) has mentioned this problem in his *Rafʿ al-ḥijāb ʿan wujūh aʿmāl al-ḥisāb* (p. 221). In the West, this problem is known as "Weighing problem of Bachet", and it was assumed to have begun with Fibonacci who gives the problem as follows: "Someone has four weiging stones with which he wants to weigh the whole pounds of his ware from 1 pound up to 40 pounds; the weights of the single weighing stones are asked for". Fibonacci gives the solution as 1, 3, 9, 27 pounds. His successors in the 16th century go up to 364 pounds for which 6 stones are necessary (Hermelink, p.51).

I have my own experience with this problem. As a schoolboy, I used to help my father in his shop in the city of Rasht, where he still works. This city is the centre of Gīlān province on the southern coast of the Caspian, in the neighbourhood of Ṭabaristān. In that shop I usually weighed different goods, so I was engaged in taking from the extant weighing stones those summing up to the desired weight. Sometimes I had leisure time to ponder over mathematical problems which attarcted me. Once I noticed that the weighing stones were made as 1, 2, 3, 5, 10 kilograms, and so on. I understood that the current decimal numeration system was effective in selecting these weights. Then I tried to ignore the privilege of number 10, and I solved the problem which led to the geometric progression with the common ratio 3. Then I tried to extend it to the case in which we are allowed to use the balance twice for each weighing. I found the weights as 1, 5, 25 and so on, and for the case with three times application of the balance for each weighing, I found the numbers, 1, 7, 49 and so on. Many years later, as a university graduate, I spoke about this point with a mathematics professor, who told me that he has heard about it as an enigma at a mathematical congress.

In fact, for the simple case given by Ṭabarī and Fibonacci, we may write:

$$w(n) - S(n-1) = S(n-1) + 1$$

$$w(n) = 2S(n-1) + 1$$

$$w(n+1) = 2S(n) + 1.$$

By subtraction, we have : $w(n+1) - w(n) = 2w(n)$, which gives :

$$\mathbf{w(n+1) = 3\,w(n)}$$

For the second case, we have :

$$w(n) - 2S(n-1) = 2S(n-1) + 1$$

$$w(n) = 4S(n-1) + 1$$

$$w(n+1) = 4S(n) + 1.$$

Again, by subtraction : $w(n+1) - w(n) = 4(n)$, and $\mathbf{w(n+1) = 5\,w(n)}$.

It would be easy to show that for k times application of the balance for each weighing, it is enough to have the weighing stones in a geometrical progression beginning with 1 and with the common ratio $(2k+1)$.

Problem 49 is an engima using the *abjad* system which assigns numerical values to the letters of the Arabic alphabet, also used in Persian. The enigma asks for a name containing "1/5 and 1/10 of two similar letters". Ṭabarī demonstrates that the two similar letters are *mīm* (m) which stands for 40 in *abjad*. Then 1/5 and 1/10 of 40 are 8 and 4, which correspond to *ḥā* (h) and *dāl* (d), respectively. Then the name is found to be Mohammad (mḥmd). The process is of course non-unique, but other alternatives do not lead to a real name.

Problem 50 is a similar enigma which leads to the name *ʿAlī*. These are the earliest Persian enigmas for the names of persons. Using the *abjad* system for making such enigmas for personal names and for important dates became quite popular in the following centuries up to the 18th century, but later it lost its original importance. For important events like somebody's death or the completion of an important construction, a poem was composed and usually the last line indicated the date implicitly, through the sum of the numerical values of the letters it contained. Some extremely elaborated and complicated examples of this "art" exist. For example, on the occasion of repairing the mausoleum of Our Lady Maʿṣūma in Qum in 1218 A.H., Mohammad Sadeq Nateq Esfahani composed a poem which was later called *Qaṣīda-ye muʿjiziyya* (The miraculous elegy). It contained 124 lines, the total numerical values of each line amounting to 1218 (Nakhjavāni, pp.576-85).

Problem 54 is as follows : Three men ate some bread together. One of them had brought 3 loaves of bread, the other 2 loaves, and the third one, nothing. Therefore, the third one paid 5 **dirhams.** How should they distribute it between them? The problem is interesting, because initially one thinks distributing 5 **dirhams** by the ratio of 3 to 2, but actually we should take into account the bread eaten by the first and second men themselves. So we may write:

$$3 + 2 = 5,\ 5{:}3 = 5/3,\ 3 - 5/3 = 4/3,\ 2 - 5/3 = 1/3.$$

Then the 5 **dirhams** should be distributed by the ratio of 4 to 1.

In a Persian treatise on recreational mathematics entitled *Laṭāʾif al-ḥisāb*

("Fine points of arithmetic"), by Quṭb al-Dīn Lāhījī (d. about 1090/1678-9), the unique manuscript of which is in the Holy Shrine Library in Mashhad as MS 5609, a similar problem is given (fol. 20r). Here, instead of 3 and 2 loaves, the first two men have 5 and 3 loaves. Again, it is discussed that the 8 **dinars** paid by the third man should be distributed in the ratio of 7 to 1, and not 5 to 3. Following this, the manuscript contains another problem of the same structure, given through a story with real historical figures (fol. 20v): At the time of Ḥasan Ṣabbāḥ the famous leader of Isma'ili sect, and Khwāja Nizām al-Mulk, the Grand Vizir of the Seljukid Sultan Malikshah, the King ordered 500 *mans* of marble to be brought from Aleppo to Isfahan. The requested marble was transported by two camel drivers, one having 4 and the other 6 camels. Each camel driver had 500 *mans* as his own luggage. When they finished their work , the King paid 1000 **dinars**. Khwāja Nizām al-Mulk distributed the money between them, giving 600 and 400 dinars to the owners of and 4 camels, respectively. Then Ḥasan Ṣabbāḥ informed the King that the money was not distributed justly. He argued as follows: The total load carried by the camels weighed 1500 *mans;* so each camel carried 150 *mans,* the share of 4 camels being 600 *mans,* of which 500 was the camel driver's own luggage. The share of 6 camels was 900 *man*, again 500 of which was the second camel driver's own luggage. So the money should be divided by the ratio of (900-500) to (600-500), or the ratio of 400 to 100, or 4 to 1. The King agreed with him, and ordered to redistribute the money in the ratio of 4 to 1.

The version with 5 and 3 loaves of bread is also attributed to Ḥaẓrat ʿAlī, the first imam of the Shi'ites. In some sources e.g., *Dhakhāir al-uqbā* (cairo ed., p. 84) by Aḥmed b. 'Abd-Allāh al-Shafiʿī al-Ṭabarī (d. 694/1294-5), it is mentioned as a clever judgement by Ḥaẓrat ʿAlī.

Jacob I. Perelman, the famous Russian recreational mathematician who died during the siege of Stalingrad in the 2nd World War, has given a version of this problem in his book book on mathematical entertainment, entitled *Living Mathematics* (pp. 5-6). In his version, three travelers stayed in a room with a fireplace. One of them had brought 5 pieces of firewood, and the other one 3 pieces. The third one paid them 8 units of money. How should the first two divide it between them?

In a book on history, entitled *Nāsekh al-tawārīkh*, by Mohammad Taqi Sepehr (d. 1297/1879-80), in the volume on Ḥaẓrat ʿAlī's manners, there is another story (p. 857) about a clever judgement by Ḥaẓrat 'Ali, with a mathematical point:Three men had a problem in dividing 17 camels among themselves, because they wanted to give 1/2 to the first one, but 17 is not divisible by 2. The share of the next one was 1/3, but 17 is not divisible by 3, and the third 1/9 while 17 is not divisible by 9. They asked Ḥaẓrat ʿAlī to help them. He said: " Do you agree if I add one of my

* For a similar problem of dividing 19 cows in the ratios 1/2, 1/4, 1/5, see **Bhavan's Journal,** 36 (6), 1989, p.85, with sanskrit manuscript as source (cf. GB, 11, p.101). — Editor (RCG)

camels to your 17 camels?" They accepted. Then the total number was 18. He gave 9 camels to the first man, 6 camels to the second, 2 camels to the third one. Then he kept the remaining camel for himself*. Another version of this problem in which 39 camels are to be divided in the ratios 1/2, 1/4, 1/8, and 1/10 is given by Karl Menninger in his book entitled *Ali Baba und die neunddreissig Kamele* (pp.9-14).

REFERENCES

Abu Mansūr (Abd al-Qāhir b) Ṭāhir al-Baghdādi, *al-Takmila fi 'l-ḥisāb*, ed. A.S. Saidan, Institute of Arab Manuscripts, Kuwait, 1985.

Bahā' al-Din Āmili, *Riāẓiāt*, ed. Jalāl Shawqi, Aleppo, 1976.

Bhāskara II *Algebra, with arithmetic and mensuration, from the Sanskrit of Brahmagupta and Bhāskara*, tr. H.T. Colebrooke, London, 1817, repr. Wiesbaden, 1973.

M. Gardner, *Mathematical puzzles of Sam Loyd*, New York, 1959.

H. Hermelink, *Proceeding of the first international symposium for the history of Arabic science*, Vol. 2, ed. A.Y. al-Hassan et al, Aleppo, 1978.

Ibn al-Bannā, *Raf'al-ḥijāb 'an wujūh a'māl al-ḥisāb*, ed. M. Aballagh, Fez, 1994 (in Arabic).

Ghiyāth al-Din Jamshid al-Kāshi, *Miftāḥ al-ḥisāb*, ed. A.S. Damardāsh and M.Ḥ. al-Ḥafni al-Shaykh, Cairo, 1967.

Karl Menninger, *Ali Baba und die neunddreissig Kamele*, 10. Auflage, Köln, 1978.

H.Nakhjavāni, *Mawād-al-tawārikh*, Tehran 1964 (in Persian).

J. I. Perelman, *Zhivaya matematika* ("Living mathematics"), Moscow, 11th ed., 1978 (in Russian).

Qian Baocong (ed.), *Suan jing shi shu* (Ten mathematical classics), Peking, 1963.

U. Rebstock, *Rechnen im islamischen Orient*, Darmstadt, 1992.

C. A. Storey, *Persian literature*, Vol. II, Part 1, London, 1972.

Muhammad b. Ayyūb Ṭabari, *Miftāḥ al-mu'āmilāt*, ed. M.-A. Riyāḥi, Tehran, 1970 (in Persian).

J. Tropfke, *Geschichte der Elementarmathematik*, Band I, Berlin & New York, 1980.

"Science is a human activity embedded in the historical process, and works that significantly advanced the understanding of nature in their own age are perceived as major achievements."

"It was Newton who taught modern science the very possibility of a mathematical science of motion through physical media."

—Richard S. Westfall (1994)